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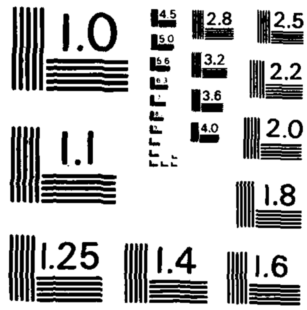

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CORRELATION STRUCTURE IN ITERATED
FARLIE-GUMBEL-MORGENSTERN DISTRIBUTIONS

by

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ABSTRACT

In this note we determine the natural parameter space of the iterated FGM distribution, and thereby show that the maximum correlation is higher than what was previously known. We also show that one single iteration can result in tripling the covariance for certain marginals. Finally, we show that there exist no marginals for which the single iteration will bring about higher negative correlation.

1. INTRODUCTION

The bivariate Farlie-Gumbel-Morgenstern (FGM) system of distributions [see Johnson and Kotz (1975)] has joint cumulative distribution function of the form

$$H(x,y) = F(x)G(y)[1 + \alpha\overline{F}(x)\overline{G}(y)] \quad , \tag{1}$$

where F and G are the marginal cumulatives, $\overline{F} = 1 - F$, $\overline{G} = 1 - G$ and α is a real number so that H is a bona fide distribution function.

The FGM distribution has a considerable appeal in model building since it provides a convenient way to construct a joint distribution having specified marginals. For details on applications, see Conway (1983) and Schucany, Parr and Boyer (1978). The usefulness of FGM, however, was marred by the fact that it is restricted to describe relatively weak dependence. As was pointed out by Schucany et al. (1978), if X and Y are bivariate H with absolutely continuous marginals then the correlation coefficient ρ between X and Y can never exceed 1/3.

Various attempts to enlarge the family have been made. Johnson and Kotz (1977) proposed the following "iterated" generalization of (1):

$$\begin{aligned} H &= FG\{1 + \alpha_1\overline{F}\overline{G} + \alpha_2 FG\overline{F}\overline{G} + \ldots + \alpha_k (FG)^{[\frac{k}{2}]}(\overline{F}\overline{G})^{[\frac{k+1}{2}]}\} \\ &= FG + \sum_{j=1}^{k} \alpha_j (FG)^{[\frac{j}{2}]+1} (\overline{F}\overline{G})^{[\frac{j+1}{2}]} \quad . \end{aligned} \tag{2}$$

Clearly, H has marginals F and G. We shall call (2) the "k-1 fold iteration." In particular, the bivariate FGM with a single iteration (k=2) will be written as

$$H = FG + \alpha FG\overline{F}\overline{G} + \beta(FG)^2\overline{F}\overline{G} \quad . \tag{3}$$

Note that since $F^k\overline{F}^{n-k} = \binom{n}{k}^{-1}(F_{kn}-F_{k+1\,n})$, where $F_{kn} = \sum_{j=k}^{n}\binom{n}{j}F^j\overline{F}^{n-j}$ is the cumulative of X_{kn}, the k-th smallest order statistic of a sample of size n from F, we can alternatively express (2) in terms of the difference of the F_{kn}'s:

$$H = FG + \alpha_1\binom{2}{1}^{-2}(F_{12}-F_{22})(G_{12}-G_{22}) + \alpha_2\binom{3}{2}^{-2}(F_{23}-F_{33})(G_{23}-G_{33}) + \ldots \quad . \tag{4}$$

If F and G have densities, so would H:

$$h = fg + \alpha_1\binom{2}{1}^{-2}(f_{12}-f_{22})(g_{12}-g_{22}) + \alpha_2\binom{3}{2}^{-2}(f_{23}-f_{33})(g_{23}-g_{33}) + \ldots \quad . \tag{5}$$

If E(X) and E(Y) exist, so would $E(X_{kn}) \equiv \mu_{kn}$ and $E(Y_{kn}) \equiv \nu_{kn}$, hence

$$\begin{aligned} E(XY) = E(X)E(Y) &+ \alpha_1\binom{2}{1}^{-2}(\mu_{12}-\mu_{22})(\nu_{12}-\nu_{22}) \\ &+ \alpha_2\binom{3}{2}^{-2}(\mu_{23}-\mu_{33})(\nu_{23}-\nu_{33}) + \ldots \quad , \end{aligned} \tag{6}$$

$$\mathrm{Cov}(X,Y) = \alpha_1\binom{2}{1}^{-2}(\mu_{12}-\mu_{22})(\nu_{12}-\nu_{22}) + \alpha_2\binom{3}{2}^{-2}(\mu_{23}-\mu_{33})(\nu_{23}-\nu_{33}) + \cdots , \quad (7)$$

$$= \alpha_1(\mu_{22}-\mu_{11})(\nu_{22}-\nu_{11}) + \alpha_2(\mu_{33}-\mu_{22})(\nu_{33}-\nu_{22}) + \cdots \quad , \quad (8)$$

last equality following from recursive formula for expected values of order statistics. Note that $\mathrm{Cov}(X,Y)$ depends on F and G only thru μ_{nn} and ν_{nn}, $n=1,2,\ldots,k+1$.

EXAMPLE 1. Specializing (3) to $F = G =$ Uniform $(0,1)$, we see that $\mu_{kn} = \nu_{kn} = k/(n+1)$, $\sigma_x^2 = \sigma_y^2 = 1/12$,

$$\mathrm{Cov}(X,Y) = \alpha/36 + \beta/144 \quad (9)$$

$$\rho = \alpha/3 + \beta/12 \quad . \quad (10)$$

EXAMPLE 2. Again for FGM with one iteration (3), let $F = G =$ Normal $(0,1)$, $\mu_{11} = \nu_{11} = 0$, $\mu_{22} = \nu_{22} = \pi^{-\frac{1}{2}}$, $\mu_{33} = \nu_{33} = 3/(2\sqrt{\pi})$,

$$\mathrm{Cov}(X,Y) = \rho = \pi^{-1}\alpha + (4\pi)^{-1}\beta \quad . \quad (11)$$

2. NATURAL PARAMETER SPACE

The natural parameter space Ω of (2) consists of all $(\alpha_1,\alpha_2,\ldots,\alpha_k)$ such that (2) represents a bona fide cumulative. For $k=1$ case, Ω

has been determined by Cambanis (1977). In particular, for absolutely continuous F and G, $\Omega = \{\alpha : |\alpha| \le 1\}$.

For $k \ge 2$ the structure of Ω is complicated and is difficult to describe in closed forms. Unfortunately, the usefulness of the FGM family (2) depends on how rich the family is. In other words, how big is Ω ?

In this note we determine Ω for $k = 2$ and for absolutely continuous F and G .

LEMMA 1. For absolutely continuous F and G, the natural parameter space Ω of (2) is convex.

PROOF. Let $0<p<1$, $\underline{\alpha} = (\alpha_1, \ldots, \alpha_k) \in \Omega$ and $\underline{\alpha}^* = (\alpha_1^*, \ldots, \alpha_k^*) \in \Omega$. By definition, $h_{\underline{\alpha}}(x,y) \ge 0$, $h_{\underline{\alpha}^*}(x,y) \ge 0$ a.e. It follows easily that $h_{p\underline{\alpha}+(1-p)\underline{\alpha}^*}(x,y) = ph_{\underline{\alpha}}(x,y) + (1-p)h_{\underline{\alpha}^*}(x,y) \ge 0$ a.e. □

For $F = G =$ Uniform $(0,1)$, let us write the density of (3) as

$$h(x,y) = 1 + \alpha(1-2x)(1-2y) + \beta xy(2-3x)(2-3y) , \quad 0 \le {x \atop y} \le 1 , \tag{12}$$

and define

$$\Lambda = \{(\alpha,\beta) : |\alpha| \le 1,\ \alpha + \beta \ge -1,\ \beta \le \frac{3-\alpha+\sqrt{9-6\alpha-3\alpha^2}}{2}\} . \tag{13}$$

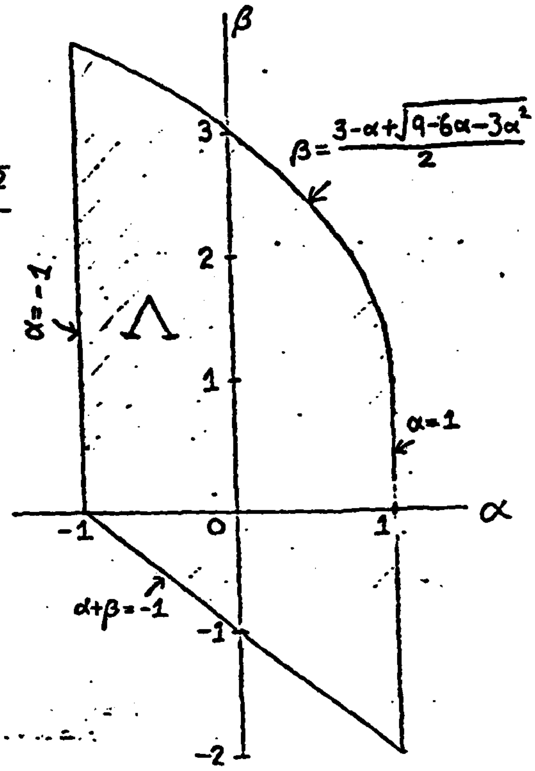
It is easy to verify that

$$\min_x h(x,0) \geq 0 \implies |\alpha| \leq 1$$

$$h(1,1) \geq 0 \implies \alpha+\beta \geq -1$$

$$\min_x h(x,1) \geq 0 \implies \beta \leq \frac{3-\alpha+\sqrt{9-6\alpha-3\alpha^2}}{2}$$

Hence $\Lambda \supseteq \Omega$.



In order to prove $\Lambda \subseteq \Omega$ let us first observe that

(i) Λ is the convex hull of $\Lambda^* \cup \{(1,-2),(-1,0)\}$, where Λ^* is the north-east boundary of Λ :

$$\Lambda^* = \{(\alpha,\beta) : -1 \leq \alpha \leq 1 \ , \ \beta = \frac{3-\alpha+\sqrt{9-6\alpha-3\alpha^2}}{2}\} \tag{14}$$

and (ii)

$$(\alpha,\beta) \in \Lambda^* \implies \alpha^2 - \beta(3-\alpha) + \beta^2 = 0 \ . \tag{15}$$

In view of the convexity of Ω, it suffices to show

$$\{(1,-2),(-1,0)\} \subseteq \Omega \tag{16}$$

and

$$\Lambda^* \subseteq \Omega \ . \tag{17}$$

(16) is easily verified. To prove (17) let us note that if $(\alpha,\beta) \in \Lambda^*$ then $h_{\alpha\beta}$ is nonnegative on the boundary of unit square:

$$h_{\alpha\beta}(x,y) \geq 0 \qquad x=0 \ \text{ or } \ x=1 \ \text{ or } \ y=0 \ \text{ or } \ y=1 \ . \tag{18}$$

Let $(\alpha,\beta) \in \Lambda^*$ and

$$y \leq u(\alpha) \equiv \begin{cases} \dfrac{\alpha+\beta+\sqrt{\alpha^2-\alpha\beta+\beta^2}}{3\beta} & \alpha \geq 0 \qquad (19) \\[2ex] \dfrac{2\beta-\alpha+\sqrt{\alpha^2+2\alpha\beta+4\beta^2}}{6\beta} & \alpha \leq 0 \ . \qquad (20) \end{cases}$$

The y-section of h, a quadratic function of x, is either concave or monotone on $(0,1)$, thus attains its minimum at the end point $x=0$ or 1, which is known to be nonegative by (18). Hence, in order to ensure $h(x,y) \geq 0$ we hereafter restrict our attention to $y > u(\alpha)$ and, by symmetry, to $x > u(\alpha)$.

For $y > u(\alpha)$, $\min_x h(x,y)$ is attained at

$$x = x_y \equiv \frac{\beta y(3y-2)-\alpha(2y-1)}{3\beta y(3y-2)} \quad .$$

Notice that for $\alpha \geq 0$, $x_y \leq \frac{2}{3} \leq u(\alpha)$, a case already disposed of. Hereafter we assume $\alpha < 0$ (thus $u(\alpha)$ is given by (20)). We leave the reader to check, making use of (15), the following:

$$\begin{aligned} -3\beta y(3y-2)h(x_y,y) &= 9\beta^2y^4 - 6\beta(2\beta-\alpha)y^3 \\ &\quad + (4\alpha^2+4\beta^2-7\alpha\beta-9\beta)y^2 + (6\beta+2\alpha\beta-4\alpha^2)y + \alpha^2 \\ &= \frac{1}{3}(y-1)[(3y-2)Q_\alpha(y) - \alpha^2] \quad , \end{aligned} \tag{21}$$

where

$$Q_\alpha(y) \equiv 9\beta^2y^2 + 3\beta(2\alpha+\beta)y + \alpha^2 \quad . \tag{22}$$

$Q_\alpha(y)$ is strictly increasing, strictly positive on $y > 0$, thus $(3y-2)Q_\alpha(y) > 0$ and is strictly increasing on $y > \frac{2}{3}$. In order to show LHS(21) < 0 for $y > u(\alpha)$ it suffices to show

$$(3u-2)Q_\alpha(u) - \alpha^2 > 0 \quad . \qquad (u \equiv u(\alpha)) \tag{23}$$

But (20) $\Rightarrow$

$$6\beta u^2 - 2(2\beta-\alpha)u - \alpha = 0 \quad . \tag{24}$$

(24) + (20) $\Rightarrow$

$$\text{LHS}(23) = -\frac{3}{2}\alpha(3\beta u-\beta+\alpha)$$

$$= -\frac{3}{4}\alpha(\alpha+\beta+\sqrt{\alpha^2+2\alpha\beta+4\beta^2}) > 0 \quad ,$$

completing the proof that $\Lambda^* \subseteq \Omega$. $\therefore \Lambda = \Omega$. for the uniform marginals case.

To prove $\Lambda = \Omega$ for arbitrary absolutely continuous F and G, let us notice that $h_{\alpha\beta}(x,y) = f(x)g(y)h^*_{\alpha\beta}(x,y)$, where

$$h^*_{\alpha\beta}(x,y) = 1+\alpha[1-2F(x)][1-2G(y)] + \beta F(x)G(y)[2-3F(x)][2-3G(y)] \quad .$$

Thus it is equivalent to the problem of showing

$$h^*_{\alpha\beta}(\xi,\eta) \geq 0 \qquad 0 \leq {\xi \atop \eta} \leq 1 \quad ,$$

reducing the problem to the uniform marginal case.

3. POSITIVE DEPENDENCE

What is the maximum dependence attainable in the FGM family (2)? Clearly if we allow F and G to be discrete, even in the noniterated case (1), it is possible to attain $\max\limits_{\alpha} \rho = 1$.

For absolutely continuous case, the problem was studied by Kotz and Johnson (1977) and Schucany et.al. (1978), both assuming $|\alpha_1| \leq 1$, $|\alpha_2| \leq 1,\ldots,|\alpha_k| \leq 1$ and $k \to \infty$.

We shall restrict our consideration to one iteration, absolutely continuous case (12), with finite σ_x and σ_y,

$$\{h_{\alpha\beta} : (\alpha,\beta) \in \Omega\} .$$

As was pointed out in (7), for given F and G, Cov(X,Y) depends only on μ_{kn} and ν_{kn}, which in turn dictates the optimal choice of α and β.

$$\rho = \frac{\text{Cov}\ (X,Y)}{\sigma_x\ \sigma_y} = \frac{\alpha}{4}\,\frac{\mu_{22}-\mu_{12}}{\sigma_x}\,\frac{\nu_{22}-\nu_{12}}{\sigma_y} + \frac{\beta}{9}\,\frac{\mu_{33}-\mu_{23}}{\sigma_x}\,\frac{\nu_{33}-\nu_{23}}{\sigma_y} . \qquad (25$$

For examples 1 and 2, the optimal choice of (α,β) is a point on the boundary set Λ^* with slope $-(\mu_{22}-\mu_{11})^2/(\mu_{33}-\mu_{22})^2$, namely, $\alpha = \frac{7}{\sqrt{13}} - 1$, $\beta = 2 - \frac{2}{\sqrt{13}}$, yielding

$$\max \rho = \begin{cases} \dfrac{\sqrt{13}-1}{6} = .43426 & \text{(Example 1)} \qquad (26) \\[2ex] \dfrac{\sqrt{13}-1}{2\pi} = .41469 & \text{(Example 2)} . \qquad (27) \end{cases}$$

For general case, to obtain a bound for (25), we note that for any F with finite σ_x,

$$\frac{\mu_{22}-\mu_{12}}{\sigma_x} \le \sqrt{\frac{4}{3}} , \quad \frac{\mu_{33}-\mu_{23}}{\sigma_x} \le \sqrt{\frac{279}{240}} \qquad (28)$$

(Ludwig (1973)). Substituting (28) into (25), we get

$$\rho \leq \frac{1}{3}\alpha + \frac{31}{240}\beta \tag{29}$$

RHS above is maximized over Λ^* at

$$\alpha = \frac{129}{\sqrt{4881}} - 1 \ , \ \beta = 2 - \frac{18}{\sqrt{4881}} .$$

Thus

$$\rho \leq \frac{1}{40}\left(\frac{1627}{\sqrt{4881}} - 3\right) = .5072 \tag{30}$$

In both Examples 1 and 2 we see that one single iteration has improved ρ from 1/3 to $(\sqrt{13}-1)/6$, and from π^{-1} to $(\sqrt{13}-1)/(2\pi)$, respectively. In both cases, the improvement is $(\sqrt{13}-3)/3$, or, a little over 30%.

Our next example shows that one single iteration could result in nearly 200% increase in covariance.

Example 3. Let $F = G$ be Pareto truncated at $t(>1)$:

$$F(x) = \frac{t}{t-1}\left(1-\frac{1}{x}\right) \qquad 1 \le x \le t \quad .$$

Then $\mu_{11} = c \ln t$, $\mu_{22} = 2c(c \ln t - 1)$, $\mu_{33} = 3c(c^2 \ln t - c - \frac{1}{2})$ where $c \equiv t/(t-1)$.

$$\frac{\mu_{22}-\mu_{11}}{\mu_{33}-\mu_{22}} \longrightarrow 1 \qquad \text{as} \quad t \to \infty \quad .$$

Thus for large t, the optimal (α,β) is close to $(0,3)$. [Note: for $\beta = \beta(\alpha)$ on A^*, $\frac{d}{d\alpha}\beta(\alpha)\Big|_{\alpha=0} = -1$]. Compared to the old FGM (1) without iteration, whose max. covariance is attained at $\alpha=1$ with (max old Cov.) $= (\mu_{22}-\mu_{11})^2$, we see that

$$\frac{(\text{max new Cov.})}{(\text{max old Cov.})} = \alpha + \beta\left[\frac{\mu_{33}-\mu_{22}}{\mu_{22}-\mu_{11}}\right]^2 \longrightarrow 3 \qquad \text{as} \quad t \to \infty \quad .$$

4. NEGATIVE DEPENDENCE

LEMMA 2. For arbitrary, nondegenerate F with finite mean, $\mu_{22} - \mu_{11} > \mu_{33} - \mu_{22}$.

PROOF. This is an immediate consequence of Theorem 1 (ii') of Kadane (1974). According to the Theorem, there exists a probability measure on the open interval (0,1) whose expectation is $(\mu_{33}-\mu_{22})/(\mu_{22}-\mu_{11})$. $\square$

A surprising consequence of the Lemma is: For the one iteration FGM (2), $\mathrm{Cov}(X,Y)$ is minimized over Ω at $(\alpha,\beta) = (-1,0)$ regardless of F and G. In other words, there exists no marginals for which the single iteration will bring about a higher negative covariance.

Another consequence of the Lemma is: Single iteration can at best triple the positive covariance, as is the case in Example 3.

Finally, we conclude with the observation that, in view of (13) and (25), maximum ρ over Ω is never attained at $\alpha = \beta = 1$ as was tacitly assumed in the literature. There always exists $(\alpha,\beta) \in \Lambda^*$, different from (1,1), having a higher ρ value.

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